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ON THE SIMPLEST INVERSE PROBLEM FOR SUMS OF SETS IN SEVERAL DIMENSIONS

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We describe the structure of $d-$dimensional sets having the smallest cardinality of the sum set. Let $K \subseteq \mathbb{R}^d$ be a finite $d-$dimensional set such that $|K+K| = (d+1)|K| - \frac{d(d+1)}{2}$. If $|K| > d+4$, then K consists of d parallel arithmetic progressions with the same common difference. We also establish the structure of K in the remaining cases $d < |K| < d+5$.

0. Notation

In what follows, K is a finite subset of $\mathbb{R}^d$—the d-dimensional Euclidean space, $\mathbb{Z}^d$ is the integer point lattice in $\mathbb{R}^d$, $\mathbb{N}$ will denote the nonnegative integers $0,1,2,\ldots$, and $|X|$ is the *cardinality* of the set X.

Let A and B be two subsets of $\mathbb{R}^d$. As usual their *sum* is defined by

$$A + B = \{x \in \mathbb{R}^d | x = a + b\ ,\ a \in A\ ,\ b \in B\}$$

and we put

$$2A = A + A.$$

The *affine dimension* $\dim A$ of a set $A \subseteq \mathbb{R}^d$ is defined as the dimension of the smallest hyperplane containing A. We denote by

$$e_0 = (0,0,\ldots,0); e_1 = (1,0,\ldots,0); \ldots; e_d = (0,0,\ldots,1)$$

the $(d+1)$ vertices of the standard d-simplex in $\mathbb{R}^d$. The *convex hull* of a set M is denoted by conv M. A vector will also be called a *point* and will be written in the form $(x_1,\ldots,x_n)$ where the x_i, $1 \le i \le n$, are the coordinates of the vector.

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1. Introduction

In the general case, for every finite sets $A, B \subseteq \mathbb{R}^d$ with $|A|, |B| \geq 2$, nothing more than the obvious $|A+B| \geq |A|+|B|-1$ can be stated, which holds with equality if and only if A and B are arithmetic progressions with the same difference.

In [1, lemma 1.14] G.A.Freiman proved the following important theorem:

$$|A+A| \geq (d+1)|A| - \frac{d(d+1)}{2}, \tag{1}$$

for every d-dimensional set A. Recently I. Z. Ruzsa [3] proved the analogue of this result for the sum $A+B$ of two finite sets. The case $B=-A$ was previously studied in [2].

The inequality (1) is tight. It holds for a union of d parallel arithmetic progressions with the same common difference. Indeed, define $A_i = \{e_i + te_d | 0 \leq t \leq k_i - 1\}$ for $0 \leq i \leq d-1$ and put $A = A_0 \cup A_1 \cup \cdots \cup A_{d-1}$. Then $|A_i| = k_i$, $\dim A = d$, $|A| = k = \sum_{0 \leq i \leq d-1} k_i$. It is not difficult to show that

$$|A+A| = (d+1)|A| - \frac{d(d+1)}{2}. \tag{2}$$

Indeed, we will prove this equality by induction on d. Put $A_* = A \setminus A_{d-1}$, $\dim A_* = d-1$. Therefore, $|A+A| = |A_* + A_*| + |A_* + A_{d-1}| + |A_{d-1} + A_{d-1}| = |A_* + A_*| + |A_*| + (d-1)(|A_{d-1}|-1) + (2|A_{d-1}|-1) = d|A_*| - \frac{(d-1)d}{2} + |A_*| + (d+1)|A_{d-1}| - d = (d+1)|A| - \frac{d(d+1)}{2}$. Equality (2) is proved and this shows that the lower bound in (1) is the best possible.

The function $|A+A|$ is an affine invariant of the set A; it follows that every d-dimensional set K which consists of d parallel arithmetic progressions with the same common difference also satisfies (2).

The inverse problem for $A+A$ is to find extremal sets such that

$$|A+A| = (d+1)|A| - \frac{d(d+1)}{2},$$
$$\dim A \geq d.$$

Note that if $\dim A = r > d$, then inequality (1) implies $|A+A| \geq (r+1)|A| - \frac{r(r+1)}{2} > (d+1)|A| - \frac{d(d+1)}{2}$. Therefore, we may assume without loss of generality that $\dim A = d$.

Not all extremal sets are disjoint unions of parallel arithmetic progressions with the same common difference. Here are some examples (see Figure 1).

For $d=2$, we take vertices of a two-dimensional simplex plus three mid-points of its three edges. Therefore

$$K_2 = \{e_0, e_1, e_2, 2e_1, e_1+e_2, 2e_2\}, |K_2| = 6 \text{ and } |K_2+K_2| = 15 = 3|K_2| - 3.$$

For $d>2$, we increase the dimension of K_2 adding points one-by-one. More precisely, we embed K_2 into $\mathbb{R}^d$ and put

$$K_d = K_2 \cup \{e_3, e_4, \ldots, e_d\}.$$

Then the elements of $K_d + K_d$ are elements of $K_2 + K_2$, the points $e_i + e_j$, for $3 \le i, j \le d$ and those of $K_2 + e_i$. Hence

$$|K_d + K_d| = 15 + \frac{(d-1)(d-2)}{2} + 6(d-2) = (d+1)|K_d| - \frac{d(d+1)}{2}.$$

We shall prove that these examples are the only examples of extremal sets that *are not* unions of d parallel arithmetic progressions with the same common difference.

2. Main Result

Let $d \ge 2$. The following theorem describes the structure of d-dimensional sets having the smallest cardinality of the sum-set.

Theorem. *Let $K \subseteq \mathbb{R}^d$ be a finite set such that*

(3) $$\dim K = d,$$

(4) $$|K+K| = (d+1)|K| - \frac{d(d+1)}{2}.$$

Then the following cases are possible:

(i) K lies on d parallel lines and consists of d parallel arithmetic progressions with the same common difference.

(ii) $K = \{v_0, v_1, \ldots, v_d\} \cup \{2v_1, v_1 + v_2, 2v_2\}$, where $v_i, 0 \le i \le d$, are the vertices of a d-dimensional simplex. ∎

Let us mention that the result, for $d=2$, is a particular case of Theorem 1.17 in [1].

Remarks.

(a) If K satisfies (i), we will say that K has *the standard structure*. In this case there is a supporting $(d-1)$-dimensional hyperplane $\mathcal{L}$ for $\operatorname{conv}(K)$ such that

$$|K \cap \mathcal{L}| = d, \ \dim(K \cap \mathcal{L}) = d-1.$$

(b) If K satisfies (ii), we will say that K has *the nonstandard structure*. In this case there is $v \in K$ such that $K_0 = K \setminus \{v\}$ and $\{v\}$ lie on two parallel hyperplanes and K_0 has the nonstandard structure in dimension $d-1$. Moreover, there is no supporting hyperplane $\mathcal{L}$ for $\operatorname{conv}(K)$ such that $|K \cap \mathcal{L}| = d$. Indeed, there are $d+1$ supporting hyperplanes having the intersection numbers given by: $d+1, d+1, d+1, d+3, \ldots, d+3$, respectively.

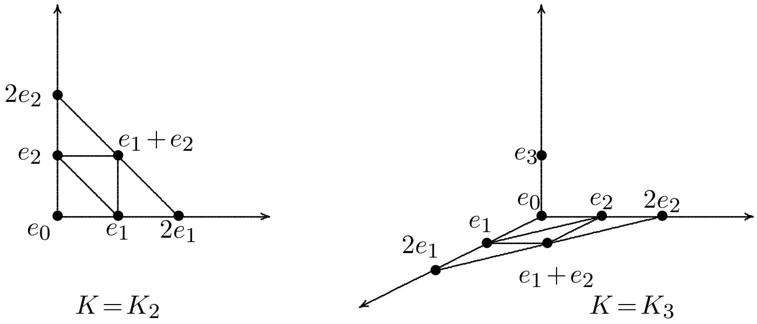


Figure 1. The nonstandard structure for $d=2$ and $d=3$

3. Proof

Let $v_0=(v_{01},\ldots,v_{0d})$ be a vertex of the convex polytope spanned by the set K. Let

$$K_0 = K \setminus \{v_0\}.$$

There are two possibilities $\dim K_0=d-1$ or $\dim K_0=d$.

Case a. Suppose that K_0 is contained in a $(d-1)$-dimensional hyperplane $\mathcal{P}$. Then by (1) we obtain

$$|K+K| \geq 1+|v_0+K_0|+|K_0+K_0| \geq 1+|K_0|+d|K_0|-\frac{(d-1)d}{2}$$
$$= (d+1)|K|-\frac{d(d+1)}{2}.$$

By hypothesis (4) it follows that $|K_0+K_0|=d|K_0|-\frac{(d-1)d}{2}$. Thus, K_0 is a $(d-1)$-dimensional finite set with minimal doubling.

If K_0 has the standard structure, then K_0 consist of $d-1$ parallel arithmetic progressions with the same common difference. Therefore, the set K has the standard structure.

If K_0 has the nonstandard structure, then $|K_0|=(d–1)+4$, $|K|=1+|K_0|=d+4$, and K has the nonstandard structure too.

Case b. Now suppose that $\dim K_0=d$ and consider the smallest polytope containing K_0. Let the hyperplane $\mathcal{L}$ be its $(d-1)$-dimensional face such that the point v_0 and the set K_0 lie in distinct half-spaces with the common boundary $\mathcal{L}$.

Since $\dim(K_0\cap\mathcal{L})=d-1$, we have $|K_0\cap\mathcal{L}|\geq d$. Then

$$|K+K| \geq |K_0+K_0|+|v_0+(K_0\cap\mathcal{L})|+1$$
$$\geq (d+1)|K_0|-\frac{d(d+1)}{2}+|K_0\cap\mathcal{L}|+1 \geq (d+1)|K|-\frac{d(d+1)}{2}.$$

By hypothesis (4) it follows that

$$|K_0 + K_0| = (d+1)|K_0| - \frac{d(d+1)}{2} \tag{5}$$

and

$$|K_0 \cap \mathcal{L}| = d. \tag{6}$$

We apply the induction hypothesis for K_0. Note that (i) $\mathcal{L}$ is a face of the convex hull of K_0, (ii) $\dim(K_0 \cap \mathcal{L}) = d-1$, (iii) $|K_0 \cap \mathcal{L}| = d$, which gives that K_0 has the standard structure; K_0 lies on d parallel lines $\ell_0, \ell_1, \ldots, \ell_{d-1}$ and consists of d parallel arithmetic progressions with the same common difference w.

After an affine isomorphism of $\mathbb{R}^d$ we may suppose that

$$\mathcal{L} = (x_d = 0), \tag{7}$$

$$K_0 \subseteq (x_d \geq 0), \tag{8}$$

$$K_0 \cap \mathcal{L} = \{e_0, e_1, \ldots, e_{d-1}\}, \tag{9}$$

$$v_0 \in (x_d < 0). \tag{10}$$

Case b.1. $w_d = \langle e_d, w\rangle = 0$. If $w = (w_1, \ldots, w_d)$ lies in $\mathcal{L}$, there is no loss in generality in assuming that

$$w = e_1. \tag{11}$$

In other words, the direction of every ℓ_i is given by e_1. (We use (9) and the standard structure of K_0.)

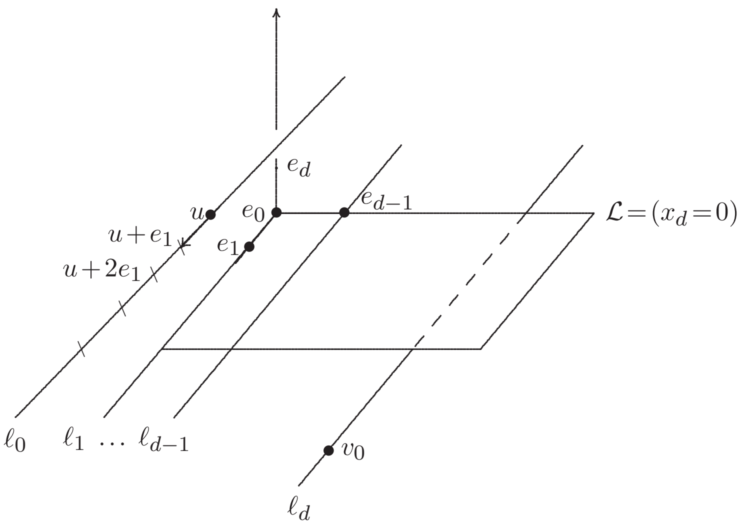


Figure 2. K consists of $d+1$ affinely independent points $\{e_0, \ldots, e_{d-1}, v_0\}$ and a finite arithmetic progression $\{u, u+e_1, u+2e_1, \ldots\}$

Take another point of K_0,

$$u \in K_0 \setminus \{e_0, e_1, \ldots, e_{d-1}\} \tag{12}$$

such that K lies on $d+1$ parallel lines $\ell_0, \ell_1, \ldots, \ell_d$ defined by

$$\begin{aligned}&\ell_1 = \{te_1 \mid t \in \mathbb{R}\}, \quad K \cap \ell_1 = \{e_0, e_1\},\\ &\ell_i = e_i + \ell_1, \quad K \cap \ell_i = \{e_i\}\ , \quad 2 \le i \le d-1,\\ &\ell_0 = u + \ell_1, \quad K \cap \ell_0 = \{u + re_1 \mid 0 \le r \le k-d-2,\ r \in \mathbb{Z}\},\end{aligned}$$

and

$$\ell_d = v_0 + \ell_1, \quad K \cap \ell_d = \{v_0\},$$

and on three parallel hyperplanes

$$\begin{aligned}&\mathcal{L}_1 = (x_d = u_d)\ ,\ |K \cap \mathcal{L}_1| = |K| - (d+1),\ K \cap \mathcal{L}_1 = K \cap \ell_0,\\ &\mathcal{L} = (x_d = 0)\ ,\ K \cap \mathcal{L} = \{e_0, e_1, \ldots, e_{d-1}\},\\ &\mathcal{L}_{-1} = (x_d = v_{0d})\ ,\ K \cap \mathcal{L}_{-1} = \{v_0\}.\end{aligned}$$

See Figure 2. Remark that e_d may not belong to K. After an affine isomorphism of $\mathbb{R}^d$ we may suppose that

$$v_0 = -e_d. \tag{13}$$

First we determine the intersection of the line ℓ_0 with the hyperplane $(x_1 = 0)$. We will prove that $\ell_0 = (P + e_d) + \ell_1$, where $P \in \{e_0, e_2, \ldots, e_{d-1}\} + \{e_0, e_2, \ldots, e_{d-1}\}$. Moreover, we will show that K has at most three points on ℓ_0 and $K \cap \ell_0 \subseteq \{P + e_d, P + e_d + e_1, P + e_d + 2e_1\}$.

Take an arbitrary point $x = (x_1, \ldots, x_d) \in K \cap \mathcal{L}_1 = K \cap \ell_0$.

Note that

$$x + v_0 \in (K \cap \mathcal{L}) + (K \cap \mathcal{L}) = \{e_0, e_1, \ldots, e_{d-1}\} + \{e_0, e_1, \ldots, e_{d-1}\}. \tag{14}$$

(Otherwise, the removal of v_0 from K would reduce the cardinality of $K+K$ by at least $d+2$: the $d+1$ usual ones $2v_0$, $v_0 + e_i$, $(0 \le i \le d-1)$ and $x + v_0$. Thus $|K+K| \ge (d+2) + |K_0 + K_0|$ which, in view of (5), contradicts the minimality of $|K+K|$).

Since ℓ_0 is parallel to e_1 and $x = e_i + e_j - v_0 = e_i + e_j + e_d \in \ell_0$ (for some i, j $0 \le i, j \le d-1$) then

$$\ell_0 \cap (x_1 = 0) = P + e_d, \quad \text{where} \quad P \in \{e_0, e_2, \ldots, e_{d-1}\} + \{e_0, e_2, \ldots, e_{d-1}\}, \tag{15}$$

$$\ell_0 = (P + e_d) + \ell_1, \tag{16}$$

and for every $x \in K \cap \mathcal{L}_1 = K \cap \ell_0 \subseteq (K \cap \mathcal{L}) + (K \cap \mathcal{L}) + e_d$ we have

$$x_i \in \mathbb{N}\ , \quad x_1 + \cdots + x_{d-1} \le 2\ , \quad x_d = 1. \tag{17}$$

This gives

$$K \cap \ell_0 \subseteq \{P + e_d, P + e_d + e_1, P + e_d + 2e_1\}, \tag{18}$$

$$|K \cap \ell_0| \le 3, |K| \le d + 4. \tag{19}$$

We complete the proof of case b.1. by studying three simple situations.

(i) If $P=e_i+e_j$ with $2\le i,\ j\le d-1$, then K has the *standard structure* . Indeed, in view of (15), (16), (17), (18) we obtain that $\ell_0\cap(x_1=0)=e_i+e_j+e_d, \ell_0=(e_i+e_j+e_d)+\ell_1$,

$$K\cap\ell_0=\{e_i+e_j+e_d\},\quad |K|=d+2, \tag{20}$$

and so K lies on the following two parallel hyperplanes:

$$K\cap(x_1=0)=\{e_0,-e_d;e_2,\dots,e_{d-1};e_i+e_j+e_d\},$$
$$K\cap(x_1=1)=\{e_1\}.$$

We are in the case described in Figure 3.

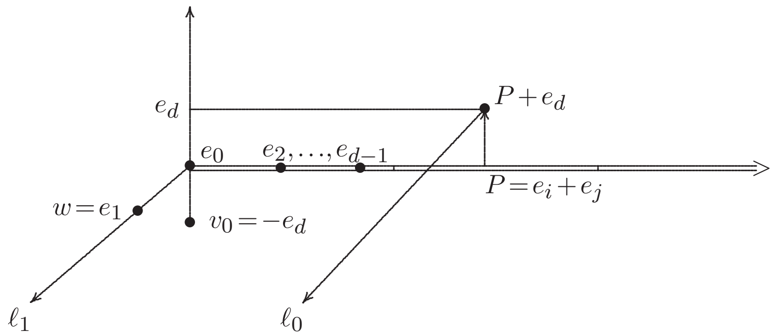


Figure 3. (see case (i):$P=e_i+e_j;2\le i,j\le d-1$) K has the standard structure

If $i=j$, then the points $e_i+e_j+e_d=2e_i+e_d$, $e_i,-e_d$ are collinear and so K lies on d parallel lines and has the standard structure: d arithmetic progressions with the common difference $w_0=e_i+e_d$.

If $i\ne j$, then the 4 points $\{e_i+e_j+e_d,e_j\}\cup\{e_i,-e_d\}$ define two parallel lines and it is clear that $K\cap(x_1=0)$ lies on $d-1$ parallel lines. In conclusion K itself lies on d parallel lines and has the standard structure: d arithmetic progressions with the common difference $w_0=e_i+e_d$.

(ii) If $P=e_0+e_i=e_i$ with $2\le i\le d-1$, then K has also the *standard structure*. Indeed, from (15),(16),(17),(18) it follows that $\ell_0\cap(x_1=0)=e_i+e_d, \ell_0=(e_i+e_d)+\ell_1$,

$$K\cap\ell_0\subseteq\{e_i+e_d\ ,\ e_i+e_d+e_1\},\quad |K|\le d+3, \tag{21}$$

and so K lies on the following two parallel hyperplanes:

$$\{e_0,-e_d,e_2,\dots,e_{d-1}\}\subseteq K\cap(x_1=0)\subseteq\{e_0,-e_d,e_2,\dots,e_{d-1},e_i+e_d\},$$
$$e_1\in K\cap(x_1=1)\subseteq\{e_1,e_i+e_d+e_1\}.$$

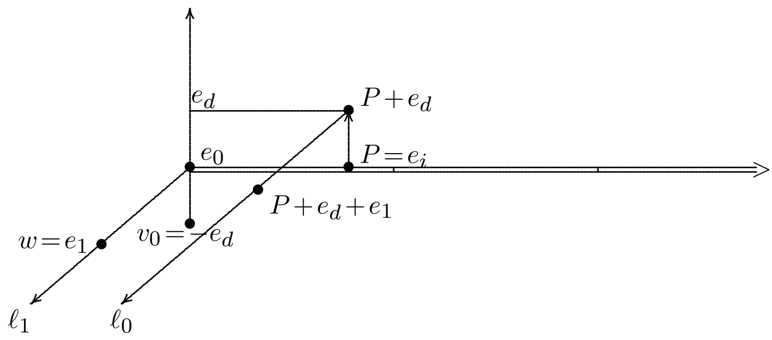


Figure 4. (see case (ii): $P=e_i, 2\le i\le d-1$) K has the standard structure

We are in the case described in Figure 4.

The 6 points $\{e_0, e_i+e_d\}$, $\{e_1, e_i+e_d+e_1\}$, $\{-e_d, e_i\}$ lie on three parallel lines. Therefore, K itself lies on d parallel lines and has the standard structure: d arithmetic progressions with the common difference $w_0=e_i+e_d$.

(iii) If $P=e_0$, then K has the *standard structure*, unless $|K\cap\ell_0|=3$. In this case K has the *nonstandard structure*. Indeed, in view of (15),(16),(17),(18) it follows that $\ell_0\cap(x_1=0)=e_d, \ell_0=e_d+\ell_1$,

$$K\cap\ell_0\subseteq\{e_d, e_d+e_1, e_d+2e_1\},\quad |K|\le d+4\ , \tag{22}$$

and so K lies on at most three parallel hyperplanes:

$$\{e_0, -e_d, e_2, \ldots, e_{d-1}\}\subseteq K\cap(x_1=0)\subseteq\{e_0, -e_d; e_2, \ldots, e_{d-1}; e_d\},$$
$$e_1\in K\cap(x_1=1)\subseteq\{e_1, e_d+e_1\},$$
$$K\cap(x_1=2)\subseteq\{e_d+2e_1\}.$$

If $K\cap\ell_0=\{e_d, e_d+e_1, e_d+2e_1\}$, then K has the *nonstandard structure*: in the plane π spanned by (e_1, e_d) the two-dimensional nonstandard set $\{-e_d; e_0, e_1; e_d, e_d+e_1, e_d+2e_1\}$ to whom we add $d-2$ linearly independent vectors $e_2, \ldots, e_{d-1}$. See Figure 5.

If $|K\cap\ell_0|=2$ or 1, then $K\cap\pi$ has at most 5 points and lies on 2 parallel lines; $K\setminus\pi=\{e_2, \ldots, e_{d-1}\}$ and therefore K has the standard structure: d parallel arithmetic progressions with the same common difference $w_0=e_d$, if $e_d+2e_1\notin K$, or $w_0=e_1+e_d$, if $e_d+2e_1\in K$. See Figure 6.

Remark that the case $K\cap\ell_0=\{e_d, e_d+2e_1\}$ is not allowed. Indeed, if $K\cap\ell_1=\{e_0, e_1\}$ and $K\cap\ell_0=\{e_d, e_d+2e_1\}$, then we contradict the standard structure of K_0: d parallel arithmetic progressions with the same common difference $w=e_1$. The proof of Theorem B in case b.1. is now complete.

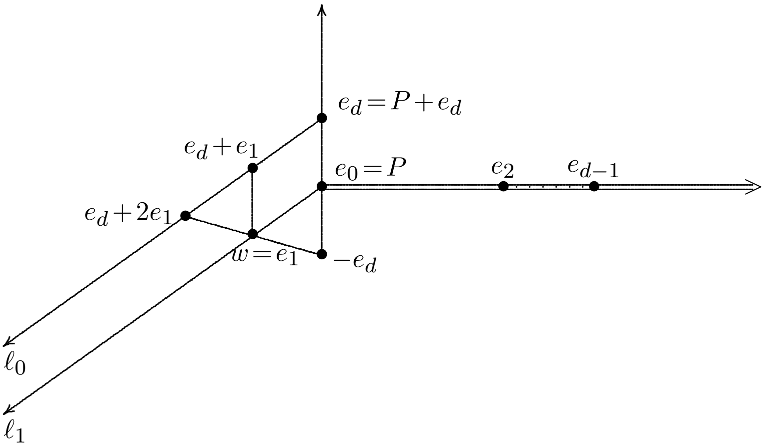
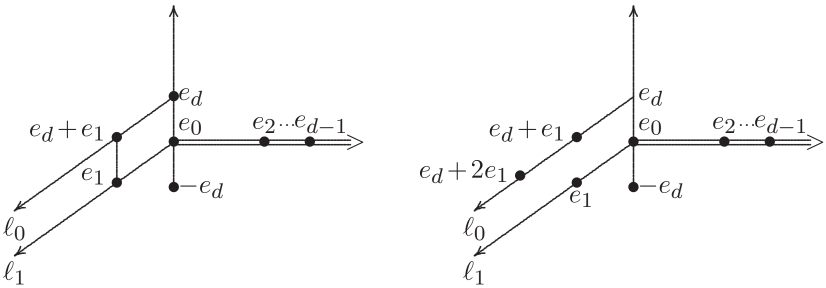


Figure 5. (see case (iii): $P = e_0$ and $|K \cap \ell_0| = 3$) K has the non-standard structure



$e_d + 2e_1 \notin K$ $\qquad$ $e_d + 2e_1 \in K$

Figure 6. K has the standard structure, if $P = e_0$ and $|K \cap \ell_0| \leq 2$

Case b.2. $w_d \neq 0$. If w doesn't lie in $\mathcal{L}$, there is no loss of generality in assuming that

$$w = e_d. \tag{23}$$

It follows that K_0 lies on d parallel lines $\ell_0 = \{te_d \mid t \in \mathbb{R}\}$, $\ell_i = e_i + \ell_0$, $1 \leq i \leq d-1$, and $K = K_0 \cup \{v_0\}$ with $v_{0d} < 0$. We are in the case described in Figure 7 and we will prove that K has *the standard structure*.

K_0 consists of d parallel arithmetic progressions with the same common difference e_d, $K_0 \subseteq (x_d \geq 0)$ and $\dim K_0 = d$. Moreover, $e_i \in K_0$ for every $0 \leq i \leq d-1$. Therefore, there is i_0, $0 \leq i_0 \leq d-1$ such that $x = e_{i_0} + e_d \in K_0$.

Note that $x + v_0 = e_{i_0} + v_0 + e_d \in \{e_0, e_1, \ldots, e_{d-1}\} + \{e_0, e_1, , \ldots, e_{d-1}\}$. (Otherwise the removal of v_0 from K would reduce the cardinality of $K+K$ by at least $d+2$: the $d+1$ usual ones $2v_0$, $v_0 + e_i, (0 \leq i \leq d-1)$ and $x + v_0$. Thus $|K+K| \geq (d+2) + |K_0 + K_0|$

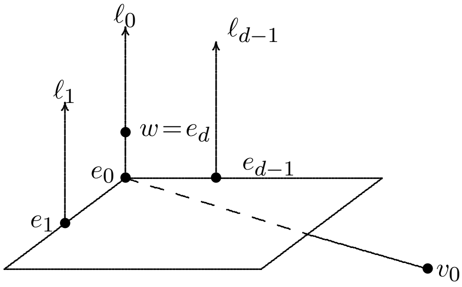


Figure 7

which contradicts the minimality of $|K+K|$, in view of (5)). For simplicity suppose $i_0=0$, which gives $x=e_d$. Up to this point we know that

$$\{e_0,e_d\}\subseteq K_0\cap\ell_0, \tag{24}$$

$$v_0\in\big(\{e_0,e_1,\ldots,e_{d-1}\}+\{e_0,e_1,\ldots,e_{d-1}\}\big)-e_d. \tag{25}$$

α) If $v_0=e_i-e_d$, with $0\le i\le d-1$, then K consists of d parallel arithmetic progressions with the same common difference $w_0=e_d$; K lies on d parallel lines $\ell_i, 0\le i\le d-1$.

β) If $v_0=(e_i+e_j)-e_d$, with $1\le i,j\le d-1$, then K lies on three parallel hyperplanes

$$\begin{aligned}&\kappa_0: x_1+x_2+\cdots+x_{d-1}=0,\ K\cap\kappa_0=K\cap\ell_0,\\&\kappa_1: x_1+x_2+\cdots+x_{d-1}=1,\ K\cap\kappa_1=K\cap\big(\ell_1\cup\cdots\cup\ell_{d-1}\big)\supseteq\{e_1,\ldots,e_{d-1}\},\\&\kappa_2: x_1+x_2+\cdots+x_{d-1}=2,\ K\cap\kappa_2=\{v_0\}.\end{aligned}$$

We will prove that the minimality of $|K+K|$ implies

$$|K\cap\kappa_1|=d-1,\ K\cap\kappa_1=\{e_1,\ldots,e_{d-1}\}\ \text{ and} \tag{26}$$

$$K=\{v_0;e_0,e_1,\ldots,e_{d-1};e_d\}=\{e_i+e_j-e_d,e_0,\ldots,e_d\}. \tag{27}$$

It follows that K has the standard structure: d parallel arithmetic progressions with the same common difference $w_0=v_0-e_i$. We complete the proof by showing that (26) and (27) are true.

If $|K\cap\kappa_1|\ge d+1$, then we would obtain $|K+K|\ge 1+|v_0+(K\cap\kappa_1)|+|K_0+K_0|\ge d+2+|K_0+K_0|$, which contradicts the minimality of $|K+K|$, in view of (5).

If $|K\cap\kappa_1|=d$, then there is only one $j_0, 1\le j_0\le d-1$ such that $K\cap\ell_{j_o}=\{e_{j_0},e_{j_0}+e_d\}$. It follows that $K\supseteq\{v_0;e_0,e_1,\ldots,e_{d-1};e_d,e_{j_0}+e_d\}$ and K lies on at least three parallel hyperplanes

$$\begin{aligned}&\mathcal{H}_0:\ x_1+x_2+\cdots+x_d=0,\ \ K\cap\mathcal{H}_0=\{e_0\},\\&\mathcal{H}_1:\ x_1+x_2+\cdots+x_d=1,\ \ K\cap\mathcal{H}_1=\{e_1,\ldots,e_d;v_0\},\dim(K\cap\mathcal{H}_1)=d-1,\\&\mathcal{H}_2:\ x_1+x_2+\cdots+x_d=2,\ \ K\cap\mathcal{H}_2\supseteq\{e_{j_0}+e_d\}.\end{aligned}$$

Put $K^* = K \setminus \{e_0\}$ and note that $\dim K^* = d$. The removal of e_0 from K would reduce the cardinality of $K+K$ by at least $d+2$. Indeed, we have $|K+K| \geq 1 + |e_0+(K\cap\mathcal{H}_1)|+|K^*+K^*| = 1+(d+1)+|K^*+K^*| = (d+2)+|K^*+K^*|$, which again contradicts the minimality of $|K+K|$.

Therefore (26) is true and

$$K \cap \kappa_1 = \{e_1,\dots,e_{d-1}\},\ K \cap \ell_i = \{e_i\},\ 1 \leq i \leq d-1,\ |K \cap \ell_0| = |K| - d.$$

The set $K \setminus \{v_0, e_0, e_1, \dots, e_{d-1}\}$ lies on $\ell_0 \cap (x_d > 0)$. It remains to show that $|K\cap\ell_0| = 2$. If the arithmetic progression on ℓ_0 has more than two elements (that is, if $e_0+2e_d \in K$), then the removal of v_0 from K reduces the cardinality of $K+K$ by at least $d+2$ elements: $2v_0, v_0+e_i, 0 \leq i \leq d-1$, and v_0+2e_d. This contradicts the minimality of $|K+K|$, in view of (5). In conclusion (27) is true. The proof is now complete. ∎

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